Recall Kähler Manifolds (M^{2n}, J, g, ω)

$$d = \partial + \bar\partial = d^{1,0} + d^{0,1} \qquad \partial^2 = \bar\partial^2 = 0$$

$$L : \Omega^* \to \Omega^{*+2}$$

$$L(\alpha) = \omega \wedge \alpha$$

ω closed $\Rightarrow [L, d] = 0$

$[L, \partial] = 0$

$[L, \bar\partial] = 0$

Adjoint $\Lambda = L^*$

Kähler Identities

$$[\partial, \Lambda] = -i\bar\partial^*$$

$$\Rightarrow [\bar\partial, \Lambda] = i\partial^*, \quad [L, \bar\partial^*] = -i\partial, \quad [L, \partial^*] = i\bar\partial.$$

Consequences: $\partial\bar\partial^* + \bar\partial^*\partial = 0$

$$\Rightarrow \Delta = \Delta_\partial + \Delta_{\bar\partial}, \quad \Delta_\partial = \Delta_{\bar\partial}$$

$$\Rightarrow \Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$$

$$H^k(M) = \bigoplus_{r+s=k} H^{r,s}_{\bar\partial}(M)$$

Lefschetz decomp: $\Omega^k = \bigoplus_{i\geq 0} L^i(P^{k-2i})$, where

$P^j = \{\alpha \in \Omega^j : \Lambda\alpha = 0\}$.

primitive j-forms.

$$H^{a,b}_\Delta = \bigoplus_{k\geq 0} L^k P^{a-k,b-k}_{\Delta=0}$$

Similarly eigenspace of Laplacian $E^{a,b}_{\lambda,\Delta} = \bigoplus_{k\geq 0} L^k P^{a-k,b-k}_{\lambda,\Delta}$

Consequences: $\lambda > 0$, $\mu^j_{\lambda,\Delta}$ = multiplicity of eigenvalue λ of Δ on j-forms

$\mu^j_{\lambda,\Delta}$ is even if j is odd (also for $\lambda = 0$).

$\mu^n_{\lambda,\Delta}$ is always even ~

$\forall\ 1 \le l \le n,\ \mu^l_{\lambda,\Delta} \ge 2\,\mu^0_{\lambda,\Delta}$

in fact <u>monotonicity</u> for $0 \le k < n$

$$\mu^k_{\lambda,\Delta} \le \mu^{k+1}_{\lambda,\Delta}$$

<u>Sasaki Manifolds</u>

odd dimensional.

(M^{2n+1}, ξ, g)

unit vector field (ξ), metric (g)

ξ Killing vector field (generates 1-parameter family of isometries)

$\xi^\flat = \xi^* = \eta$ one form

contact form:

$d\eta \leftarrow$ nondegenerate symplectic form on $D = \xi^\perp \subseteq TM^{2n+1}$ (2n-dim)

$\overset{\shortparallel}{2\omega} \leftarrow$ transverse Kähler form

$L = d\eta \wedge = \varphi \omega \wedge$

$2\omega = 2g(\cdot, \phi\,\cdot)$

ϕ transverse complex structure

$\phi^2 = \begin{cases} -I & \text{on } \xi^\perp \\ 0 & \text{on } \xi \end{cases}$.

We can divide TM into pieces

$T^{1,0,0}M = +i$ eigenspace of ϕ

$T^{0,1,0}M = -i$ eigenspace of ϕ

$T^{0,0,1}M = \text{span}\{\xi\} = 0$-eigenspace of ϕ.

Thm $\Omega^k = \bigoplus\limits_{\substack{r+s+j=k \\ 0 \le j \le 1}} \Omega^{r,s,j}$

Then $d = d^{1,0,0} + d^{0,1,0} + \underbrace{d^{0,0,1}}_{(\eta\wedge)\mathcal{L}_\xi} + \underbrace{d^{1,1,-1}}_{2L(\xi\lrcorner)}$

$$\partial = d^{1,0,0}$$
$$\bar\partial = d^{0,1,0}$$
$$\partial_c = d^{1,0,0} + \tfrac{1}{2} d^{1,1,-1}$$
$$\bar\partial_c = d^{0,1,0} + \tfrac{1}{2} d^{1,1,-1}$$

all these are differentials → Cohomology groups.

Kohn, Rossi – studied cohomology of $\bar\partial_c$ on horizontal forms
$\bar\partial$ on horizontal forms. (no η)

Schmude (physicist) – studied $\bar\partial_c$, $\Delta_{\bar\partial_c}$ as operator on all forms.
Sasaki-Einstein manifolds.

$\bar\partial = d^{0,1,0}$, $\partial = d^{1,0,0}$, $d^{0,0,1}$, $d^{1,1,-1}$

$\eta \wedge \mathcal{L}_\xi$ $\quad 2L i_\xi$

$L = (\omega\wedge) = \frac{1}{2}(d\eta)\wedge$

example $\eta = \sum x_i dy_i - y_i dx_i = \xi^b$
$d\eta = \sum 2\, dx_i \wedge dy_i$

$i_\xi = \xi\lrcorner$

$\partial = \partial_c$ on horizontal forms.

$\alpha^k = \alpha_0^k + \eta\wedge\alpha_1^{k-1}$

and diff form; no η

horizontal $\Leftrightarrow i_\xi\alpha = 0 \Leftrightarrow \alpha = \alpha_0$.

vertical $\Leftrightarrow \eta\wedge$horizontal.

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identities $[\partial, L] = 0 \quad [\bar\partial, L] = 0$
$[\partial_c, L] = [\bar\partial_c, L] = 0$. ✓ just like Kähler

Schwunde $[\partial_c, \Lambda] = -i\bar{\partial}_c^* + i(\eta_1)\Lambda - H i_\xi$

where $H = n - d_0$
p
degree of horizontal part.

however $[\bar{\partial}, \Lambda] = -i\bar{\partial}^*$

$$d = \partial_c + \bar{\partial}_c + (\eta_1)\mathcal{L}_\xi$$
$$d = \partial + \bar{\partial} + (\eta_1)\mathcal{L}_\xi + 2L i_\xi$$

Facts: $\Delta_{\partial_c} = \partial_c\partial_c^* + \partial_c^*\partial_c$

$$\Delta_{\partial_c} = \Delta_{\bar{\partial}_c} + \{(\partial_c - \bar{\partial}_c), (\eta_1)\Lambda\} - i(\bar{\partial}_c + \partial_c)i_\xi - 2iH\mathcal{L}_\xi$$

$$\Rightarrow \Delta = 2\Delta_{\bar{\partial}_c} - \mathcal{L}_\xi^2 - 2iH\mathcal{L}_\xi + 2L\Lambda + 2A(\eta_1)i_\xi + 2i((\eta_1)\bar{\partial}_c^* - \bar{\partial}_c i_\xi)$$

Using ∂ & $\bar{\partial}$

$$\Delta_\partial = \Delta_{\bar{\partial}} - 2iH\mathcal{L}_\xi$$

$$\Delta = 2\Delta_{\bar{\partial}} - \mathcal{L}_\xi^2 - 2iH\mathcal{L}_\xi + 2i\eta_1(\bar{\partial}^* - \bar{\partial}) + 2i\, i_\xi(\bar{\partial} - \partial) + 4 i_\xi \eta_1 L\Lambda + 4\eta_1 i_\xi \Lambda L.$$

Consider forms ω s.t. $\mathcal{L}_\xi \omega = iq\omega$

$q \geq 0$.

$q =$ "the charge"

We want look at eigenvalues of Δ when restricted to charge q.

$$\langle \Delta\omega, \omega\rangle = 2\langle \Delta_{\bar\partial}\omega, \omega\rangle + q^2\langle \omega, \omega\rangle + 2q\langle H\omega, \omega\rangle + 4\langle \eta\wedge\Lambda\omega, \eta\wedge\Lambda\omega\rangle + 4\langle i_\xi L\omega, i_\xi L\omega\rangle + 2\langle (N+N^*)\omega, \omega\rangle,$$ where

$N = i(\partial - \bar\partial) i_\xi$ — nilpotent

$N^2 = 0$

N: vertical $\to$ horiz

N: horiz $\to 0$.

Example: consider horizontal forms.

$$\langle \Delta\omega, \omega\rangle = 2\langle \Delta_{\bar\partial}\omega, \omega\rangle + q^2\langle \omega, \omega\rangle + 2q\langle \underset{\uparrow\, n-d_0}{H}\omega, \omega\rangle + 4\langle \Lambda\omega, \Lambda\omega\rangle$$

$$\geq q^2\langle \omega, \omega\rangle + 2q\underset{\uparrow\, n-d_0}{H}\langle \omega, \omega\rangle$$

If $d_0 \leq n$, $\Delta\omega = 0$,

Then $\Delta_{\bar\partial}\omega = 0$, $q^2\langle \omega, \omega\rangle = 0$

$\Lambda\omega = 0$

$\bar\partial$ harmonic. charge $= 0$ primitive form.

Other calculations show if α is horizontal,

& $q \neq 0$,

$\Delta_{\bar{\partial}} \alpha = 0 \Rightarrow \alpha$ primitive or basic (ie $\mathcal{L}_{\xi} \alpha = 0$)